

# Appendix to “Measuring Relations between Concepts in Conceptual Spaces”

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## Abstract

This appendix provides a mathematical proof for the formula describing the hypervolume of a hyperball in a conceptual space (Proposition 1 in the paper “Measuring Relations Between Concepts In Conceptual Spaces” by Lucas Bechberger and Kai-Uwe Kühnberger [1]).

## 1 Hyperballs under the Unweighted Metric

In general, a hyperball of radius  $r$  around a point  $p$  can be defined as the set of all points with a distance of at most  $r$  to  $p$ :

$$H = \{x \in CS \mid d(x, p) \leq r\}$$

If the Euclidean distance  $d_E$  is used, this corresponds to our intuitive notion of a ball – a round shape centered at  $p$ . However, under the Manhattan distance  $d_M$ , hyperballs have the shape of diamonds. Under the combined distance  $d_C^\Delta$ , a hyperball in three dimensions has the shape of a double cone (cf. Figure 1).

As similarity is inversely related to distance, one can interpret a hyperball in a conceptual space as the set of all points that have a minimal similarity  $\alpha$  to the central point  $p$ , where  $\alpha$  depends on the radius of the hyperball.

In this section, we assume an unweighted version of  $d_C^\Delta$ :

$$d_C^\Delta(x, y) = \sum_{\delta \in \Delta} \sqrt{\sum_{d \in \delta} |x_d - y_d|^2}$$

In order to derive a formula for the hypervolume of a hyperball under  $d_C^\Delta$ , we need to use the following three lemmata:

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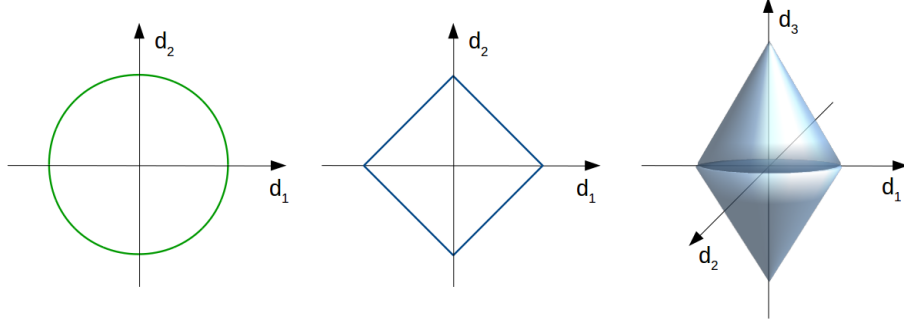


Figure 1: Left: Two-dimensional hyperball under the Euclidean metric. Middle: Two-dimensional hyperball under the Manhattan metric. Right: Three-dimensional hyperball under the combined metric (with domain structure  $\Delta = \{\delta_1 = \{d_1, d_2\}, \delta_2 = \{d_3\}\}$ ).

**Lemma 1.**

$$\int_0^{2\pi} \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2}) d\phi_1 \cdots d\phi_{n-1} = 2 \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

where  $\Gamma(\cdot)$  is Euler's Gamma function and  $n \in \mathbb{N}$ .

*Proof.*

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\pi \int_0^\pi \cdots \int_0^\pi \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2}) d\phi_1 \cdots d\phi_{n-1} \\ &= \left( \int_0^{2\pi} 1 d\phi_{n-1} \right) \left( \int_0^\pi \sin(\phi_{n-2}) d\phi_{n-2} \right) \cdots \left( \int_0^\pi \sin^{n-2}(\phi_1) d\phi_1 \right) \\ &= \left( 4 \cdot \int_0^{\frac{\pi}{2}} 1 d\phi_{n-1} \right) \left( 2 \cdot \int_0^{\frac{\pi}{2}} \sin(\phi_{n-2}) d\phi_{n-2} \right) \cdots \left( 2 \cdot \int_0^{\frac{\pi}{2}} \sin^{n-2}(\phi_1) d\phi_1 \right) \end{aligned}$$

We can now use the definition of the Beta function, which is

$$B(x, y) = 2 \cdot \int_0^{\frac{\pi}{2}} \sin^{2x-1}(\phi) \cos^{2y-1}(\phi) d\phi$$

Using  $y = \frac{1}{2}$ , we get:

$$\begin{aligned} I &= \left( 4 \cdot \int_0^{\frac{\pi}{2}} 1 d\phi_{n-1} \right) \left( 2 \cdot \int_0^{\frac{\pi}{2}} \sin(\phi_{n-2}) d\phi_{n-2} \right) \cdots \left( 2 \cdot \int_0^{\frac{\pi}{2}} \sin^{n-2}(\phi_1) d\phi_1 \right) \\ &= 2 \cdot B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot B\left(1, \frac{1}{2}\right) \cdots B\left(\frac{n-2}{2}, \frac{1}{2}\right) \cdot B\left(\frac{n-1}{2}, \frac{1}{2}\right) \end{aligned}$$

Next, we use the identity  $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$  with Euler's Gamma function  $\Gamma$  and the fact that  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ . We get:

$$I = 2 \cdot \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} \cdot \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdots \frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})} \cdot \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} = 2 \cdot \frac{\Gamma(\frac{1}{2})^n}{\Gamma(\frac{n}{2})} = 2 \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

□

**Lemma 2.** For any natural number  $j > 0$  and any  $a, b \in \mathbb{R}$ , the following equation holds:

$$\int_0^{r - \sum_{i=1}^{j-1} r_i} r_j^{a-1} \cdot \left( r - \sum_{i=1}^j r_i \right)^b dr_j = B(a, b+1) \cdot \left( r - \sum_{i=1}^{j-1} r_i \right)^{a+b}$$

*Proof.* We can make a variable change by defining  $r_j = \left( r - \sum_{i=1}^{j-1} r_i \right) \cdot z$  which gives  $dr_j = \left( r - \sum_{i=1}^{j-1} r_i \right) \cdot dz$ . This gives us:

$$\begin{aligned} & \int_0^{r - \sum_{i=1}^{j-1} r_i} r_j^{a-1} \cdot \left( r - \sum_{i=1}^j r_i \right)^b dr_j \\ &= \int_0^1 \left( r - \sum_{i=1}^{j-1} r_i \right)^{a-1} \cdot z^{a-1} \cdot \left( r - \sum_{i=1}^{j-1} r_i - \left( r - \sum_{i=1}^{j-1} r_i \right) \cdot z \right)^b \cdot \left( r - \sum_{i=1}^{j-1} r_i \right) dz \\ &= \left( r - \sum_{i=1}^{j-1} r_i \right)^{a-1+b+1} \int_0^1 z^{a-1} (1-z)^b dz = \left( r - \sum_{i=1}^{j-1} r_i \right)^{a+b} \cdot B(a, b+1) \end{aligned}$$

The last transformation uses the fact that  $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ . □

**Lemma 3.** For any natural number  $k > 0$ , any  $r_1, \dots, r_k, n_1, \dots, n_k > 0$ ,  $n = \sum_{i=0}^k n_i$ ,  $r = \sum_{i=0}^k r_i$  the following equation holds:

$$\int_0^r r_1^{n_1-1} \int_0^{r-r_1} r_2^{n_2-1} \cdots \int_0^{r - \sum_{i=1}^{k-1} r_i} r_k^{n_k-1} dr_k \cdots dr_1 = \frac{r^n}{\Gamma(n+1)} \prod_{i=1}^k \Gamma(n_i)$$

*Proof.* Using Lemma 2, we can solve the innermost integral by setting  $j = k$ ,  $a = n_k$ ,  $b = 0$ , which gives us  $B(n_k, 1) \cdot \left( r - \sum_{i=1}^{k-1} r_i \right)^{n_k}$ . Therefore:

$$\begin{aligned} I &= \int_0^r r_1^{n_1-1} \int_0^{r-r_1} r_2^{n_2-1} \cdots \int_0^{r - \sum_{i=1}^{k-1} r_i} r_k^{n_k-1} dr_k \cdots dr_1 \\ &= B(n_k, 1) \cdot \int_0^r r_1^{n_1-1} \cdots \int_0^{r - \sum_{i=1}^{k-2} r_i} r_{k-1}^{n_{k-1}-1} \cdot \left( r - \sum_{i=1}^{k-1} r_i \right)^{n_k} dr_{k-1} \cdots dr_1 \end{aligned}$$

As one can see, we can again apply Lemma 2 to the innermost integral. Repeatedly applying Lemma 2 finally results in:

$$I = B(n_k, 1) \cdot B(n_{k-1}, n_k + 1) \cdot \dots \cdot B(n_1, n_2 + \dots + n_k + 1) \cdot r^{n_1 + \dots + n_k}$$

We use that  $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$  in order to rewrite this equation:

$$I = r^{n_1 + \dots + n_k} \cdot \frac{\Gamma(n_k)\Gamma(1)}{\Gamma(n_k + 1)} \cdot \frac{\Gamma(n_{k-1})\Gamma(n_k + 1)}{\Gamma(n_{k-1} + n_k + 1)} \dots \frac{\Gamma(n_1)\Gamma(n_2 + \dots + n_k + 1)}{\Gamma(n_1 + n_2 + \dots + n_k + 1)}$$

Because  $\Gamma(1) = 1$  and because most of the terms cancel out, this reduces to:

$$I = r^{n_1 + \dots + n_k} \cdot \Gamma(n_k) \cdot \dots \cdot \Gamma(n_1) \cdot \frac{1}{\Gamma(n_1 + \dots + n_k + 1)} = \frac{r^n}{\Gamma(n+1)} \prod_{i=1}^k \Gamma(n_i)$$

□

Using these three lemmata, we can now derive the size of a hyperball in a conceptual space without domain and dimension weights:

**Lemma 4.** *The hypervolume of a hyperball with radius  $r$  in a space with the unweighted combined metric  $d_C^\Delta$  and the domain structure  $\Delta$  can be computed in the following way, where  $n$  is the overall number of dimensions and  $n_\delta$  is the number of dimensions in domain  $\delta$ :*

$$V(r, \Delta) = \frac{r^n}{n!} \prod_{\delta \in \Delta} \left( n_\delta! \frac{\pi^{\frac{n_\delta}{2}}}{\Gamma\left(\frac{n_\delta}{2} + 1\right)} \right)$$

*Proof.* The hyperball can be defined as the set of all points that have a distance of maximally  $r$  to the origin, i.e.,

$$H = \left\{ x \in CS \mid d_C^\Delta(x, 0) = \sum_{\delta \in \Delta} \sqrt{\sum_{d \in \delta} x_d^2} \leq r \right\}$$

If we define  $\forall \delta \in \Delta : r_\delta := \sqrt{\sum_{d \in \delta} x_d^2}$ , we can easily see that  $\sum_{\delta \in \Delta} r_\delta \leq r$ . The term  $r_\delta$  can be interpreted as the distance between  $x$  and the origin within the domain  $\delta$ . The constraint  $\sum_{\delta \in \Delta} r_\delta \leq r$  then simply means that the sum of domain-wise distances is less than the given radius. One can thus interpret  $r_\delta$  as the radius within domain  $\delta$ .

We would ultimately like to compute

$$V(r, \Delta) = \int \dots \int_H 1 dH$$

This integration becomes much easier if we use spherical coordinates instead of the Cartesian coordinates provided by our conceptual space.

Let us first consider the case of a single domain  $\delta$  of size  $n$ . A single domain corresponds to a standard Euclidean space, therefore we can use the standard procedure of changing to spherical coordinates (cf., e.g., [2]). Let us index the dimensions of  $\delta$  as  $d_1, \dots, d_n$ . The coordinate change within the domain  $\delta$  then looks like this:

$$\begin{aligned} x_1 &= t \cdot \cos(\phi_1) \\ x_2 &= t \cdot \sin(\phi_1) \cdot \cos(\phi_2) \\ &\vdots \\ x_{n-1} &= t \cdot \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cdot \cos(\phi_{n-1}) \\ x_n &= t \cdot \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cdot \sin(\phi_{n-1}) \end{aligned}$$

In order to switch the integral to spherical coordinates, we need to calculate the volume element. This can be found by looking at the determinant of the transformation's Jacobian matrix. The Jacobian matrix of the transformation of a single domain  $\delta$  can be written as follows:

$$J_\delta = \begin{bmatrix} \frac{\delta x_1}{\delta t} & \frac{\delta x_1}{\delta \phi_1} & \cdots & \frac{\delta x_1}{\delta \phi_{n-1}} \\ \frac{\delta x_2}{\delta t} & \frac{\delta x_2}{\delta \phi_1} & \cdots & \frac{\delta x_2}{\delta \phi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta x_n}{\delta t} & \frac{\delta x_n}{\delta \phi_1} & \cdots & \frac{\delta x_n}{\delta \phi_{n-1}} \end{bmatrix} = \begin{bmatrix} \cos(\phi_1) & -t \sin(\phi_1) & 0 & 0 & \cdots & 0 \\ \sin(\phi_1) \cos(\phi_2) & t \cos(\phi_1) \cos(\phi_2) & -t \sin(\phi_1) \sin(\phi_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cos(\phi_{n-1}) & \cdots & \cdots & \cdots & \cdots & -t \sin(\phi_1) \cdots \sin(\phi_{n-2}) \sin(\phi_{n-1}) \\ \sin(\phi_1) \cdots \sin(\phi_{n-2}) \sin(\phi_{n-1}) & \cdots & \cdots & \cdots & \cdots & t \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cos(\phi_{n-1}) \end{bmatrix}$$

The determinant of this matrix can be computed like this:

$$\det(J_\delta) = t^{n-1} \cdot \sin^{n-2}(\phi_1) \cdot \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2})$$

We can now perform the overall switch from Cartesian to spherical coordinates by performing this coordinate change for each domain individually. Let us index the Cartesian coordinates of a point  $x$  in domain  $\delta$  by  $x_{\delta,1}, \dots, x_{\delta,n_\delta}$ . Let us further index the spherical coordinates of domain  $\delta$  by  $r_\delta$  and  $\phi_{\delta,1}, \dots, \phi_{\delta,n_\delta-1}$ . Let  $k = |\Delta|$  denote the total number of domains.

Because  $x_{\delta,j}$  is defined independently from  $r_{\delta'}$  and  $\phi_{\delta',j'}$  for different domains  $\delta \neq \delta'$ , any derivative  $\frac{x_{\delta,j}}{r_{\delta'}}$  or  $\frac{x_{\delta,j}}{\phi_{\delta',j'}}$  will be zero. If we apply the coordinate change to all domains at once, the Jacobian matrix of the overall transformation has therefore the structure of a block matrix:

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix}$$

The blocks on the diagonal are the Jacobian matrices of the individual domains as defined above, and all other blocks are filled with zeroes because all cross-domain derivatives are zero. Because the overall  $J$  is a block matrix, we get that  $\det(J) = \prod_{\delta \in \Delta} \det(J_\delta)$  (cf. [4]). Our overall volume element is thus

$$\det(J) = \prod_{\delta \in \Delta} \det(J_\delta) = \prod_{\delta \in \Delta} r_\delta^{n_\delta-1} \sin^{n_\delta-2}(\phi_{\delta,1}) \sin^{n_\delta-3}(\phi_{\delta,2}) \cdots \sin(\phi_{\delta,n_\delta-2})$$

The limits of the angle integrals are  $[0, 2\pi]$  for the outermost and  $[0, \pi]$  for all other integrals. Based on our constraint  $\sum_{\delta \in \Delta} r_\delta \leq r$ , we can derive the limits for the integrals over the  $r_\delta$  as follows, assuming an arbitrarily ordered indexing  $\delta_1, \dots, \delta_k$  of the domains:

$$\begin{aligned} r_1 &\in [0, r] \\ r_2 &\in [0, r - r_1] \\ r_3 &\in [0, r - r_1 - r_2] \\ &\vdots \\ r_k &\in [0, r - \sum_{i=1}^{k-1} r_i] \end{aligned}$$

The overall coordinate change therefore looks like this:

$$\begin{aligned} V(r, \Delta) &= \int \cdots \int_H 1 \, dH \\ &= \underbrace{\int_{\phi_{1,n_1-1}=0}^{2\pi} \int_{\phi_{1,n_1-2}=0}^{\pi} \cdots \int_{\phi_{1,1}=0}^{\pi} \int_{r_1=0}^r \cdots}_{\delta=1} \underbrace{\int_{\phi_{k,n_k-1}=0}^{2\pi} \int_{\phi_{k,n_k-2}=0}^{\pi} \cdots \int_{\phi_{k,1}=0}^{\pi} \int_{r_k=0}^{r-\sum_{i=1}^{k-1} r_i}}_{\delta=k} \\ &\quad \underbrace{r_1^{n_1-1} \sin^{n_1-2}(\phi_{1,1}) \cdots \sin(\phi_{1,n_1-2})}_{\delta=1} \cdots \underbrace{r_k^{n_k-1} \sin^{n_k-2}(\phi_{k,1}) \cdots \sin(\phi_{k,n_k-2})}_{\delta=k} \\ &\quad \underbrace{dr_k d\phi_{k,1} \cdots d\phi_{k,n_k-1}}_{\delta=k} \cdots \underbrace{dr_1 d\phi_{1,1} \cdots d\phi_{1,n_1-1}}_{\delta=1} \end{aligned}$$

$$\begin{aligned}
&= \underbrace{\int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \sin^{n_1-2}(\phi_{1,1}) \cdots \sin(\phi_{1,n_1-2}) d\phi_{1,1} \cdots d\phi_{1,n_1-1}}_{\delta=1} \\
&\quad \cdots \underbrace{\int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \sin^{n_k-2}(\phi_{k,1}) \cdots \sin(\phi_{k,n_k-2}) d\phi_{k,1} \cdots d\phi_{k,n_k-1}}_{\delta=k} \\
&\quad \int_0^r r_1^{n_1-1} \cdots \int_0^{r-\sum_{i=1}^{k-1} r_i} r_k^{n_k-1} dr_1 \cdots dr_k
\end{aligned}$$

By applying Lemma 1 and Lemma 3, we can write this as:

$$\begin{aligned}
V(r, \Delta) &= \left(2 \cdot \frac{\pi^{\frac{n_1}{2}}}{\Gamma(\frac{n_1}{2})}\right) \cdots \left(2 \cdot \frac{\pi^{\frac{n_k}{2}}}{\Gamma(\frac{n_k}{2})}\right) \cdot \frac{r^n}{\Gamma(n+1)} \prod_{i=1}^k \Gamma(n_i) \\
&= \frac{r^n}{\Gamma(n+1)} \cdot \prod_{i=1}^k \left(2 \cdot \pi^{\frac{n_i}{2}} \cdot \frac{\Gamma(n_i)}{\Gamma(\frac{n_i}{2})}\right)
\end{aligned}$$

We can simplify this formula further by using the identity  $\forall n \in \mathbb{N} : \Gamma(n+1) = n!$  and the rewrite  $\prod_{i=0}^k \hat{=} \prod_{\delta \in \Delta}$ :

$$\begin{aligned}
V(r, \Delta) &= \frac{r^n}{n!} \cdot \prod_{\delta \in \Delta} \left(2 \cdot \pi^{\frac{n_\delta}{2}} \cdot \frac{(n_\delta - 1)!}{\Gamma(\frac{n_\delta}{2})}\right) = \frac{r^n}{n!} \cdot \prod_{\delta \in \Delta} \left(\frac{2}{n_\delta} \cdot n_\delta! \cdot \frac{\pi^{\frac{n_\delta}{2}}}{\Gamma(\frac{n_\delta}{2})}\right) \\
&= \frac{r^n}{n!} \cdot \prod_{\delta \in \Delta} \left(n_\delta! \cdot \frac{\pi^{\frac{n_\delta}{2}}}{\frac{n_\delta}{2} \cdot \Gamma(\frac{n_\delta}{2})}\right) = \frac{r^n}{n!} \cdot \prod_{\delta \in \Delta} \left(n_\delta! \cdot \frac{\pi^{\frac{n_\delta}{2}}}{\Gamma(\frac{n_\delta}{2} + 1)}\right)
\end{aligned}$$

The last transformation uses the fact that  $\forall x \in \mathbb{R}^+ : \Gamma(x) \cdot x = \Gamma(x+1)$ .  $\square$

## 2 Hyperballs under the Weighted Metric

We now generalize our results from the previous section from the unweighted to the weighted combined metric  $d_C^\Delta$ .

**Proposition 1.** *The hypervolume of a hyperball with radius  $r$  in a space with the weighted combined metric  $d_C^\Delta$ , the domain structure  $\Delta$ , and the set of weights  $W$  can be computed by the following formula, where  $n$  is the overall number of dimensions and  $n_\delta$  is the number of dimension in domain  $\delta$ :*

$$V(r, \Delta, W) = \frac{1}{\prod_{\delta \in \Delta} w_\delta \cdot \prod_{d \in \delta} \sqrt{w_d}} \cdot \frac{r^n}{n!} \cdot \prod_{\delta \in \Delta} \left(n_\delta! \cdot \frac{\pi^{\frac{n_\delta}{2}}}{\Gamma(\frac{n_\delta}{2} + 1)}\right)$$

*Proof.* As Gärdenfors has already argued in [3], putting weights on dimensions in a conceptual space is equivalent to stretching each dimension of the unweighted space by the weight assigned to it.

If the overall radius of a ball is  $r$ , and some dimension has the weight  $w$ , then the farthest away any point  $x$  can be from the origin on this dimension must satisfy  $w \cdot x = r$ , i.e.,  $x = \frac{r}{w}$ . That is, the ball needs to be stretched by a factor of  $\frac{1}{w}$  in the given dimension, thus its hypervolume also changes by a factor of  $\frac{1}{w}$ . A hyperball under the weighted metric is thus equivalent to a hyperellipse under the unweighted metric.

In our case, the weight for any dimension  $d$  within a domain  $\delta$  corresponds to  $w_\delta \cdot \sqrt{w_d}$ : If we look at a point  $x$  with coordinates  $(0, \dots, 0, x_d, 0, \dots, 0)$ , then  $d(0, x) = w_\delta \cdot \sqrt{w_d \cdot x_d^2} = w_\delta \cdot \sqrt{w_d} \cdot x_d$  (with  $\delta$  being the domain to which the dimension  $d$  belongs). If we multiply the size of the hyperball by  $\frac{1}{w_\delta \cdot \sqrt{w_d}}$  for each dimension  $d$ , we get:

$$\begin{aligned} V(r, \Delta, W) &= \frac{1}{\prod_{\delta \in \Delta} \prod_{d \in \delta} w_\delta \sqrt{w_d}} \cdot V(r, \Delta) \\ &= \frac{1}{\prod_{\delta \in \Delta} \prod_{d \in \delta} w_\delta \sqrt{w_d}} \cdot \frac{r^n}{n!} \cdot \prod_{\delta \in \Delta} \left( n_\delta! \cdot \frac{\pi^{\frac{n_\delta}{2}}}{\Gamma(\frac{n_\delta}{2} + 1)} \right) \end{aligned}$$

This is the hypervolume of a hyperball under the weighted combined metric.  $\square$

## References

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